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THE REALIZATION OF IMAGINARY POINTS.

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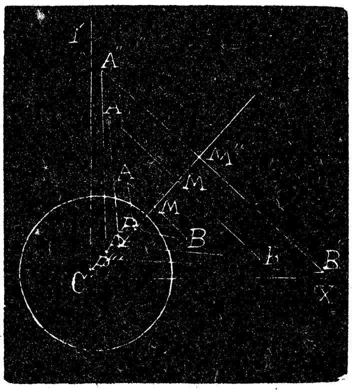
In Salmon's Conic Sections, Art. 82, the author says of an imaginary point: "It is a purely analytical conception, which we do not attempt to represent geometrically," * * * * "but attention to these imaginary points is necessary," * * * * "we shall meet with many cases in which the line joining two imaginary points is real."

It is here proposed to trace some of these imaginary points, in the hope of finding them upon the real line which joins them.

Take the points of contact of tangents to a circle from a given point. The tangents are said to be real when the point from which they are drawn is without the circle, coincident when the point is on the circle, and imaginary when the point is within the circle. It is these last tangents and their points of contact, which it is proposed to find. These last points will be found, if any where, upon the polar of the given point, which is the "real line joining the imaginary points."

The co-ordinate of contact of tangent to a circle from any point $x'y'$, are $x''=\dfrac{R^2x'\pm Ry'\sqrt{x'^2+y'^2-R^2}}{x'^2+y'^2}$, $y''=\dfrac{R^2y'\mp Rx'\sqrt{x'^2+y'^2-R^2}}{x'^2+y'^2}$.

It will be shown that these formulas may be so interpreted as to determine the points of contact of tangents, from a point *within* the circle, to a curve which may be regarded as a mere development of the circle, and that the line joining these points is the polar of the given point.



When $x'y'$ reaches the circumference the expression under the radical sign becomes $x'^2+y'^2-R^2=0$ or $x'^2+y'^2=R^2$, which may be written $R^2-(x'^2+y'^2)$. *Proceeding now with the application of the formulas we obtain the following results:

Let C be the centre of a circle, CX, CY the axes, the radius equal to 4, and the distances of the points P, P', P'' from the centre respectively 3, 2, 1. Then for P or (x^1, y^1) we have

$$x''=\frac{16\times2.12\pm4\times2.12\sqrt{16-9}}{9}$$

$=6.25$ or 1.28: $y''=1.28$ or 6.25. These determine the points A, B. The line joining these points is the polar of P, as proved by $CM=\dfrac{R^2}{CP}=\dfrac{16}{3}=5\frac{1}{3}$. For P' we have $x''=\dfrac{16\times1.4\pm4\times1.4\sqrt{16-4}}{4}$

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*By substituting the sign difference in place of the sign minus in the expression under the radical, the same result is reached.

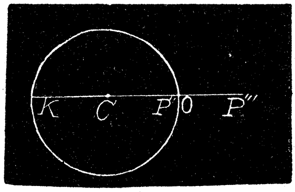
$=10.44$ or 0.75, which give the points A', B'. Joining $A'B'$ we have the polar of P' verified by $CM' = \frac{R^2}{CP'} = \frac{16}{2} = 8$. For P'' $(x'y')$ we have $x'' = 16 \times 0.707 \pm 4 \times 0.707 \sqrt{16-1} = 22.225$ or 0.369. These determine the points A'', B''. The line joining these points $A''B''$ is the polar of P'', as proved by $CM'' = \frac{R^2}{CP''} = \frac{16}{1} = 16$.

The locus of $x''y''$, while P or $x'y'$ moves along the same radius, is an equilateral hyperbola, concentric with the given circle and having its vertex at the circumference.

The proof is as follows: The co-ordinates of contact of tangents to a hyperbola, when equilateral and its semi-axes each equal to R, are $x'' = \frac{R^2x' \mp Ry'\sqrt{R^2-(x'^2-y'^2)}}{x'^2-y'^2}$ $y'' = \frac{R^2y' \pm Rx'\sqrt{R^2-(x^2-y'^2)}}{x^2-y'^2}$. CM is now the axis of $X's$, and since $P(x'y')$ moves along this line, $y'=0$. The formulas give for P, $x'=3$, $x''=5\frac{1}{3}$ or CM. $y''=\pm 3.52$ or MA and MB, which determine the points A, B. For $P'=x'=2$, $x''=8$ or CM', $y''=\pm 6.92$ or $M'A'$ and $M'B'$. For P'', $x'=1$, $x''=16$, or CM'', $y''=\pm 15.48$ or $M''A''$ and $M''B''$, all the same points as by the first formulas. Thus it appears that the *imaginary* tangents to a circle are *real* tangents to a corresponding hyperbola.

An attempt will now be made to justify the above results without recourse to the disputable expedient of changing the signs of the expression under the radical.

It is now understood that the difference between positive and negative quantities is merely the difference of counting in opposite directions, both equally real. Guided by this hint, if we let (a^2) stand for the expression under the radical sign, and consider that $(-a^2)$ is composed of factors, one of which is affected with the sign $(-)$, then, when $\sqrt{a^2} (=a\sqrt{+1})$ becomes $\sqrt{-a^2} (=a\sqrt{-1})$, we know that a factor has changed the direction in which it was before estimated, to the opposite direction. By examination we discover where and how the change took place.



In the present case it is the reversal of direction in counting from the the point $x'y'$ to the circumference. This may be seen by factoring the expression under the radical. Taking the point P''' $x'y'$ when without the circle, $x'^2+y'^2-R^2 = P'''C^2 - OC^2 = (P'''C + OC)(P'''C - OC) = P'''K \times P'''O$, both factors measured in the same direction. When we reach P'' *within* the circle, we have $P''K \times P''O$, in which $P''O$ is measured in the opposite direction and of course takes the sign $(-)$, which explains the change of sign in the radical expression $(\sqrt{-a^2})$. In other words, the instant that $x'y'$ crosses the circumference, $\sqrt{-1}$ appears. Call it then the sign of that crossing and nothing else, and at once the imaginary is divested of its badge of licensed irresponsibility and falls into the ranks of orderly realities, which alone properly constitute the exact sciences. Instead of being "an expression for an impossible operation" the $\sqrt{-1}$, in its present situation at least, calls for no operation whatever, its origin being accounted for, and its office defined just as it stands—a finger-post where before it stood an impassable barrier.

The $\sqrt{-1}$ takes its rise as a residual factor or coefficient of the term whose root has been extracted. Like the corresponding factor, $\sqrt{+1}$ it has not power to either increase or diminish the *numerical* value of the term. Why then treat it as having any numerical significance? Why not call it the co-efficient of *direction*? Whether the direction be *opposite* or *perpendicular*, say that while the symbol denotes *change* of direction the *particular* direction must in each instance, be determined by its own conditions. Although by convention (—) indicates direction opposite to (+), that will not prevent $\sqrt{-1}$ from indicating the same when the conditions of its appearance distinctly point that way, provided this use of the symbol involves no inconsistency with its other uses.

The conditions in the present example may be restated thus: Of the elements which enter into the radical expression, viz. R and $x'y'$, the latter is referred to the limits of the former, that is to the circumference and the centre. While $x'y'$ is without the circle, both limits lie in the same direction from it. After $x'y'$ enters the circle, the limits (centre and circumference) lie in opposite directions from it. And this change is signalized by the appearance of $\sqrt{-1}$.

If C be a fixed point and $x'y'$ a moving point, approaching C on a right line from an infinite distance, then by the application of the first formulas above, the tangents will generate, by a *continuous movement*, first the semi-circumference and then the hyperbola; showing the hyperbola to be an inverse continuation of the circle, or the circle turned inside out.

When $x'y'$ reaches the limit C the tangents find their true limits in, the asymtotes, and coincide with the axes. The momentary coincidence of the tangents at the other limit—the circumference—only marks the transition from one phase of their work to the other.

The tangents, moving in obedience to a uniform law, would naturally be expected, after generating the semi-circumference, not to wander off into "untraceable labyrinths," but to follow some plain path, especially as the line joining their (imaginary) points of contact, may, at any stage, be definitely located.

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